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NUMERICAL AND ANALYTICAL SOLUTIONS FOR CONCENTRATION POLARIZATION IN HYPERFILTRATION WITHOUT AXIAL FLOW

F. BELLUCCI and A. POZZI*
Istituto di Principi di Ingegneria Chimica, Facoltà di Ingegneria, Università di Napoli, Italy

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Abstract—The non-linear one-dimensional unsteady problem describing the separation of the components of a solution by an ideal semipermeable membrane is studied.

By means of a suitable application of the Laplace transform with respect to the space variable, the non-linear partial differential equation governing the system has been transformed into an integral equation.

Numerical and approximate analytical solutions of the governing integral equation are presented.

The results obtained with approximate analytical solutions have been compared with the exact numerical solutions and satisfactory agreement has been found.

NOMENCLATURE

a, constant, see equation (22);
b, constant, see equation (22);
c, concentration;
D, diffusion coefficient;
erf, error function: $\operatorname{erf} x = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-x^2}\,dx$;
erfc, complementary error function: $\operatorname{erfc}(x) = 1-\operatorname{erf}(x)$;
g, the Laplace transform of r;
k, membrane constant;
L, Laplace operator;
P, pressure;
r, dimensionless concentration, see (4);
r_0, dimensionless concentration for $y = 0$;
R, operator, see equation (9);
t, time variable;
v, velocity;
V, dimensionless velocity, see (4);
x, space variable;
y, dimensionless space variable, see (4).

Greek letters

β, constant, see equation (10);
γ, dimensionless concentration: $\gamma = r_0+1$;
Γ, gamma function;
δ, ratio between osmotic and hydrostatic pressure;
ΔP, hydrostatic pressure difference across the membrane;
$\Delta\pi^*$, difference between the osmotic pressure of the salt solution and that of the effluent;
τ, dimensionless time variable, see (4);
τ_k, asymptotic time, see equations (24), (25);
$\bar{\varepsilon}$, error percent.

*Istituto di Matematica, Facoltà di Ingegneria, Università di Napoli.

INTRODUCTION

RECENTLY an increasing interest has been devoted to the membrane separation technique for producing separation and purification without phase changes; in fact no heat addition is required and the process can be considered isothermal. In particular, hyperfiltration membranes already have many applications in the industrial field such as, for example, in seawater desalination. In recent years, these techniques have also been extended to the separation and concentration of solutions of macromolecules [1, 3, 6]. Knowledge of the fluid-dynamic field of some basic reverse osmosis systems improves the process control.

In this paper an indefinite membrane is considered and the unsteady flow is assumed to be one dimensional in a direction perpendicular to the membrane. The rejection coefficient is unitary, i.e. we assume that the concentration of the solute on the lower side of the membrane is zero; this is the case of an ideal efficiency. In practice the rejection coefficient of a real membrane can reach values very close to one (0·99). The solution density is considered to be constant; this hypothesis is very important as, in fact, the system of equations is uncoupled and only diffusion equations are to be solved.

Even under these assumptions, the problem is very difficult. If the mass flow rate can be considered constant the diffusion equation becomes linear and can be solved by a Laplace transform technique [8]. This case occurs when the ratio between the osmotic and hydrostatic pressure, δ is zero.

In [7] and [9] some approximate solutions are presented; they refer to values of δ which are very small and very close to one. In [4] the integral method was applied.

In this paper numerical and analytical solutions are presented. Both analyses are based on an integral form of the diffusion equation; in this way it was possible to obtain in short computer times accurate numerical results and approximate solutions in all fields of interest.

BASIC EQUATIONS AND BOUNDARY CONDITIONS

The equations governing an incompressible, isothermal, non-homogeneous, one-dimensional flow are the continuity, diffusion and momentum equations; the unknown functions are the velocity $v(x,t)$, the pressure $p(x,t)$ and the concentration $c(x,t)$. The first equation is satisfied by any function depending only on t, $v = v(t)$, whereas the second and the third ones give the concentration $c(x,t)$ and the pressure $p(x,t)$ respectively. The pertinent boundary conditions are

$$c(x,0) = c(\infty,t) = c_0 \quad (1)$$

$$v(0,t)c(0,t) = Dc_x(0,t) \quad (2)$$

$$\Delta\pi^* - \Delta P = kv(0,t) \quad (3)$$

where D is the binary diffusion coefficient, $\Delta\pi^*$ and ΔP are the osmotic and hydrostatic pressure respectively and k is the membrane constant. The osmotic pressure can be written as: $\Delta\pi^* = \pi_0^* c(0,t)/c_0$ where π_0^* and c_0 are reference values, i.e. π_0^* is the osmotic pressure at the concentration c_0.

From equation (3) the velocity v, equal to $v(0,t)$, can be obtained. By introducing the following dimensionless quantities

$$\delta = \pi_0^*/\Delta P; \quad \tau = \left(\frac{\Delta P}{k}\right)^2 \frac{t}{D}; \quad y = x\frac{\Delta P}{kD}; \quad r = \frac{c}{c_0} - 1; \quad V = \delta r(0,\tau) + \delta - 1 \quad (4)$$

the diffusion equation and the relative boundary conditions are

$$r_\tau + Vr_y = r_{yy} \quad (5)$$

$$r(y,0) = r(\infty,\tau) = 0; \quad r_y(0,\tau) = V[r(0,\tau)+1]. \quad (6)$$

This problem has been solved in closed form [8] only for $\delta = 0$, i.e. for constant flow rate; in particular for $r(0,\tau)$ one has

$$r(0,\tau) = \operatorname{erf}[\tau^{1/2}/2] + (\tau/2)\operatorname{erfc}(-\tau^{1/2}/2) + (\tau/\pi)^{1/2}\exp(-\tau/4). \quad (7)$$

In general, for $\delta \neq 0$, equations (5) and (6) represent a non-linear problem with respect to τ; therefore the application of the standard Laplace transform techniques with respect to τ results very hard.

INTEGRAL FORM OF THE DIFFUSION EQUATION

The Laplace transform, with respect to y, does not directly lead to the solution because it requires two conditions at $y = 0$. However, in this way, it is possible to obtain an integral equation suitable for analytical and numerical purposes. By putting $L(r) = g(p,\tau)$, from the equations (5) and (6) one has

$$g_\tau + g(pV - p^2) = -pr(0,\tau) - V. \quad (8)$$

The formal solution of this equation is

$$g = -\int_0^\tau (v + pr_0)\exp\{[p^2 + p(1-\delta)] \times (\tau - s) - p\delta R\}\,ds \quad (9)$$

where $r_0 = r(0,\tau)$ and

$$R(\tau,s) = \int_s^\tau r_0(w)\,dw.$$

Riemann's inversion formula and equation (9) give the following expression for r

$$r(y,\tau) = \frac{1}{2\sqrt{\pi}}\int_0^\tau \{(1-\delta)(1+r_0/2) - \delta r_0[1 + R/2(\tau-s)] + yr_0/2(\tau-s)\}(\tau-s)^{-1/2}\,e^{-\beta^2(\tau-s)}\,ds \quad (10)$$

where $2\beta(\tau,s,y) = 1 - \delta + (y - \delta R)/(\tau - s)$. This equation does not yet supply r since r_0 is an unknown function. This function can be obtained by evaluating equation (10) at $y \to 0$. Thus one has

$$r_0 = \frac{1}{\sqrt{\pi}}\int_0^\tau \{(1-\delta)(1+r_0/2) - r_0\delta[1 + R/2(\tau-s)]\} \times (\tau-s)^{-1/2}\,e^{-\beta_0^2(\tau-s)}\,ds \quad (11)$$

where $\beta_0 = \beta(\tau,s,0)$.

Equation (11) constitutes a condition of compatibility and it is an integral equation for r_0. When $\delta = 0$, equation (11) is linear and can be easily solved to obtain the expression (7) by Laplace transform with respect to τ. Once r_0 is known, equation (10) gives $r(y,\tau)$.

NUMERICAL SOLUTIONS

Numerical solutions of the differential equations (5) and (6) can be obtained by the usual procedures. This was performed for $0 \leqslant \tau \leqslant 100$. Solutions for higher values of τ require prohibitive computer times (several hours on an IBM 360/44 computer). On the other hand, from equations (10) and (11) very accurate results were obtained for $0 \leqslant \tau \leqslant 1000$ in few minutes.

To solve equation (11) the interval $(0,\tau)$ was divided by n parts: in any sub-interval τ_{j-1}, τ_j we assume (i) $\beta_0 = \text{const.}$ (ii) a linear expression for r_0 and therefore a quadratic expression for R. With these assumptions in any sub-interval, only I_n integrals are to be evaluated with

$$I_n = \pi^{-1/2}\int (\tau-s)^{n-3/2}\exp[-\beta_0^2(\tau-s)]\,ds \qquad n = 0,1,2\ldots.$$

These integrations can be performed in closed form in terms of error function and their derivatives. It follows that equation (11) can be written as

$$r_0 = S_0 + S_1 r_0 + S_2 r_0^2 \quad (12)$$

where S_i are known quantities.

The solutions of the equation (12) were compared with the exact analytical solution for $\delta = 0$; the results are given in Fig. 1 and show a good accuracy for all the considered values of τ. Therefore satisfactory numerical solutions can be obtained by the proposed procedure.

At this point an exact series expansion for r_0 is presented that is very suitable for small values of τ. Approximate solutions in different fields of interest will also be given.

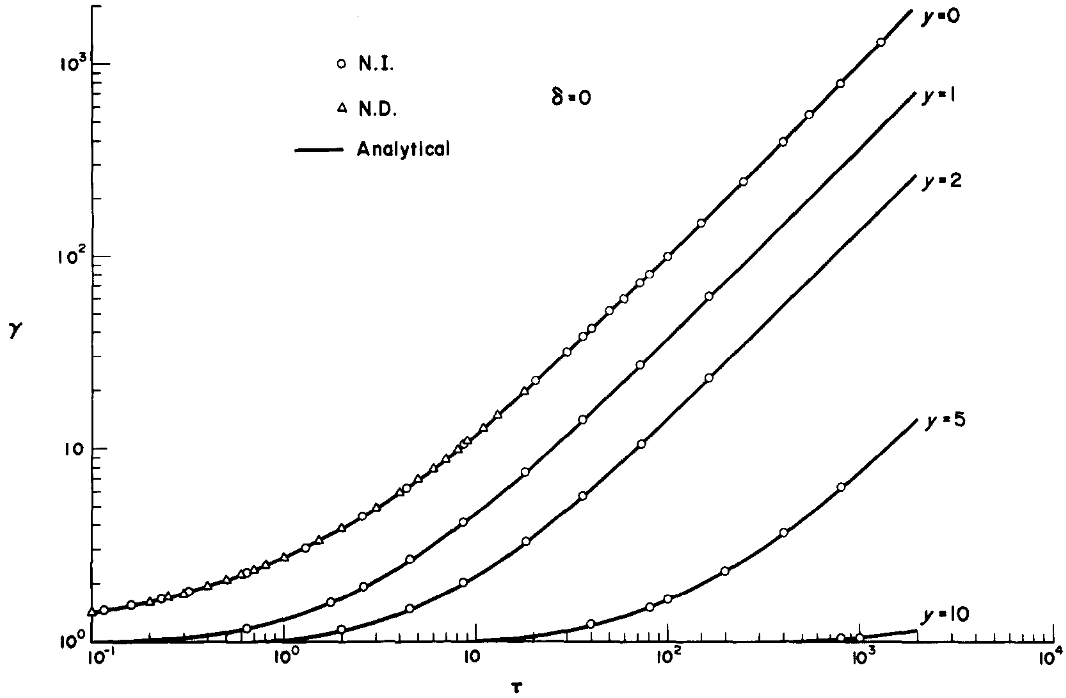

FIG. 1. Comparison between the analytical and numerical solutions for $\delta = 0$. ○, N.I.: numerical results obtained by means of integral equation (10); △, N.D.: numerical results obtained by means of differential equation (5).

SERIES EXPANSION FOR r_0

The function r_0 can be expanded in series of the variable $\tau^{1/2}$ by putting

$$r_0 = \sum_{i=1}^{\infty} c_i \tau^{i/2}. \tag{13}$$

To obtain the c_i coefficients the right side of equation (11) is integrated by parts to give

$$r_0 = \pi^{-1/2}\int_0^{\tau} \exp(-z^2/4)[1-\delta + r_0(1-2\delta) - \delta r_0^2] \times (\tau - s)^{-1/2}\,\mathrm{d}s - \int_0^{\tau} r_0' \operatorname{erf}\frac{z}{2}\,\mathrm{d}s \tag{14}$$

where $z = 2\beta_0(\tau - s)^{1/2}$.

Series expansions for the exponential function and error function give

$$\exp(-z^2/4) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\left(\frac{z}{2}\right)^{2i}; \qquad \operatorname{erf}\frac{z}{2} = \frac{2}{\pi^{1/2}}\sum_{i=0}^{\infty}\frac{(-1)^i (z/2)^{2i+1}}{i!(2i+1)}. \tag{15}$$

Moreover by expanding R in Taylor series with initial point $s = \tau$ one has

$$R = \sum_{i=0}^{\infty} \frac{r_0^i (\tau - s)^{i+1}(-1)^i}{(i+1)!}; \qquad z = \sum_{i=0}^{\infty} \frac{V^i (-1)^{i+1}(\tau - s)^{i+\frac{1}{2}}}{(i+1)!}. \tag{16}$$

The derivatives of r_0 are

$$r_0^{(i)} = \sum_{h=1}^{\infty} c_h \frac{h}{2}\left(\frac{h}{2}-1\right)\cdots\left(\frac{h}{2}-i+1\right)\tau^{h/2-i}.$$

It follows that

$$z = \sum_{i=0}^{\infty} d_i (\tau - s)^{i+\frac{1}{2}}; \quad d_i = \sum_{h=0}^{\infty} c_{i,h}\tau^{h/2-i} \tag{17}$$

where

$$c_{i,h} = \frac{(-1)^{i+1}(h/2)!}{(i+1)!(h/2-i)!}\delta c_h \quad \text{for} \quad h>0$$

$$c_{i,0} = 0 \quad \text{for} \quad i>0; \qquad c_{0,0} = 1-\delta.$$

Substitution of equations (15)–(18) into equation (11) and Cauchy rule in its generalized form for the product of series (see Appendix), lead to

$$c_n = \sum_{J=0}^{n-1}\sum_{m=m_1}^{i} \frac{G_m d_{J,m}^{(n)}}{(J-m)!(-4)^{J-m}} + 2\sum_{J=1}^{n-1}\sum_{m=m_2}^{J} \frac{m c_m D_{J,m}^{(n)}}{(-4)^{J-m+1}(J-m)![2(J-m)+1]} \tag{19}$$

where

$$d_{J,m}^{(n)} = \sum_{i=0}^{\infty} c_{i,k_1}^{2(J-m)} I_{m/2,\,J-m+i-\frac{1}{2}}$$

$$D_{J,m}^{(n)} = \sum_{i=0}^{\infty} c_{i,k_1}^{2(J-m)+1} I_{(m/2)-1,\,J-m+i+\frac{1}{2}}$$

$$c_{i_n,k}^{(n)} = \sum_{i_{n-1}=0}^{i_n}\sum_{h=0}^{k} c_{i_{n-1},h}^{(n-1)} c_{i_n - i_{n-1},\,k-h}$$

$$c_{i1,k}^{(1)} = c_{i1,k}; \quad c_{i0,k}^{(0)} = 0 \quad \text{for} \quad \begin{cases} k>0 \\ i_0 > 0 \end{cases} \text{and} \quad c_{0,0}^{(0)} = 1$$

$$I_{\alpha,j} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\alpha+1)\Gamma(j+1)}{(\alpha+j+2)}$$

$$G_0 = 1-\delta; \quad G_i = (1-2\delta)c_i - \delta Q_i;$$

$$Q_J = \sum_{m=1}^{J} c_m c_{J-m}$$

m_1 is the highest value between 0 and $2j-n+1$
m_2 is the highest value between 1 and $2j-n+1$

$$k_1 = n+m-2j-1.$$

Equation (19) furnishes very simple expressions for the first coefficients; in fact one has

$$c_1 = \frac{2}{\sqrt{\pi}}(1-\delta); \quad c_2 = (1-\delta)(1-3\delta)/2.$$

APPROXIMATE SOLUTIONS FOR r_0

From equation (11) one can obtain different types of approximate solutions.

(a) *Linearized solution*

By MacLaurin expansion of $r_0(\tau, \varepsilon)$, with $\varepsilon = (1-\delta)/\delta$, considering the first term only one has: $r_0 \cong \varepsilon r_{01}(\tau)$. Direct substitution of this expression into equation (11), via simple manipulations gives

$$r_{01} = \frac{1}{\sqrt{\pi}} \int_0^\tau (\tau-s)^{-1/2}(1-r_{01})\,\mathrm{d}s.$$

This integral equation provides the required solution, valid for $\delta - 1$, in the form

$$r_0 = \varepsilon(1-\exp(\tau)\operatorname{erfc}[\tau^{1/2}]). \tag{20}$$

(b) *Asymptotic solution*

When $\tau \to \infty$ the equation (11) can be simplified. In fact, it is

$$\lim_{\tau\to\infty} R/(\tau-s) = r_0(\infty) = (1-\delta)/\delta.$$

It must be noted that the asymptotic expression for r_0 corresponds to a vanishing value of the velocity V. Assuming this limiting value for $R/(\tau-s)$, equation (11) becomes

$$r_0 = \frac{1}{\sqrt{\pi}} \int_0^\tau (\tau-s)^{-1/2}(1-\delta-\delta r_0)\,\mathrm{d}s$$

and gives the following expression for r_0

$$r_0 = \frac{1-\delta}{\delta}[1-\exp(\delta^2\tau)\operatorname{erfc}(\delta\sqrt{\tau})]. \tag{21}$$

(c) *Initial solutions*

When $\tau \to 0$, $R/(\tau-s) \to 0$, i.e. for very small values of τ, $R/(\tau-s)$ in the equation (11) can be neglected. Therefore $r_0 = r_{01}$, where

$$r_{01} = \frac{(1-\delta)}{a^2-b}[a - b^{1/2}\operatorname{erf}(b\tau)^{1/2} - a\exp[(a^2-b)\tau]\operatorname{erfc}(a\sqrt{\tau})] \tag{22}$$

for $b \neq a^2$.

For $b = a^2$ one has

$$r_{01} = \frac{1-\delta}{2a}[\operatorname{erf}(a\sqrt{\tau}) - 2a^2\tau\operatorname{erfc}(a\sqrt{\tau}) + 2a^2(\tau/\pi)^{1/2}\exp(-a^2\tau)] \tag{22 bis}$$

where: $a = (3\delta-1)/2$; $b = (1-\delta)^2/4$.

This solution can be improved for small, but not very small values of τ.

Because $0 \leqslant R/(\tau-s) \leqslant r_0(\tau)$, $R/(\tau-s)$ can be approximated by $r_0(s)$; moreover for small values of s it is $r_0 \cong 2\pi^{-1/2}(1-\delta)s^{1/2}$. Assuming $r_0 R/(\tau-s) = 4(1-\delta)^2 s/\pi$ and solving equation (11), one has $r_0 = r_{01} + r_{02}$, where

$$r_{02} = -\frac{2\delta(1-\delta)}{\pi(a^2-b)b^{1/2}} \times \left\{(\tfrac{1}{2} - b\tau - \frac{a^2}{(a^2-b)}\operatorname{erf}(b\tau)^{1/2} - (b\tau/\pi)^{1/2}\,\mathrm{e}^{-b\tau} + \frac{ab^{1/2}}{a^2-b} \times [1-\exp[(a^2-b)\tau]\operatorname{erfc}(a\sqrt{\tau}) + ab^{1/2}\tau\right\} \tag{23}$$

and r_{01} is given by equation (22).

ASYMPTOTIC TIME

We define as asymptotic time τ_k, the time after which the relative difference between the asymptotic value and the interfacial concentration r_0 is less than k, i.e., for $\tau > \tau_k$ one has $r_0 > (1-k)r_0(\infty) = (1-k)(1-\delta)/\delta$. For very small values of δ, τ_k is obtained from the solution of [9].

By considering the asymptotic value of the error function one finds

$$(1-\delta)(1-k)/\delta = [1-(2\delta\tau)^{-1/2}](1-\delta)/\delta$$

and consequently

$$\tau_k = 1/(2\delta k^2) \quad \text{for} \quad 0 < \delta < 0{\cdot}2. \tag{24}$$

For the other values, i.e. for $0{,}2 \leqslant \delta \leqslant 1$ better accuracy is obtained from equation (21) and the asymptotic evaluation gives

$$(\tau_k = 1/(\pi\delta^2 k^2). \tag{25}$$

APPROXIMATE SOLUTIONS FOR $r(v, \tau)$

For small values of y, the following representation can be used

$$r(y,\tau) = \sum_{i=0}^{\infty} y^i f_i(\tau).$$

By substituting this expression in the equations (5) and (6) and by equating the coefficients of similar powers of y one has

$$f_0 = r_0; \quad f_1 = V(1+r_0);$$
$$f_{i+2} = f_i'/(i+2)(i+1) + Vf_{i+1}/(i+2).$$

Therefore the solution is found in the term of r_0.

At high values of y an asymptotic solution can be evaluated. In fact equation (10) can be written as

$$r(y,\tau) = \frac{1}{4\sqrt{\pi}} \int_0^\tau r_0 y(\tau-s)^{-3/2}\exp[-y^2/4(\tau-s)]\,\mathrm{d}s. \tag{26}$$

Also in this case the solution can be obtained in term of r_0; therefore, for any range of the variable, one has the appropriate expression of r_0 and the corresponding solution of equation (26).

For instance, let us consider equation (13); by simple manipulation one has:

$$r(y,\tau) = \sum_{h=1}^{\infty} c_h \Gamma(h/2+1)(4\tau)^h i^h \operatorname{erfc}(y/2\sqrt{\tau})$$

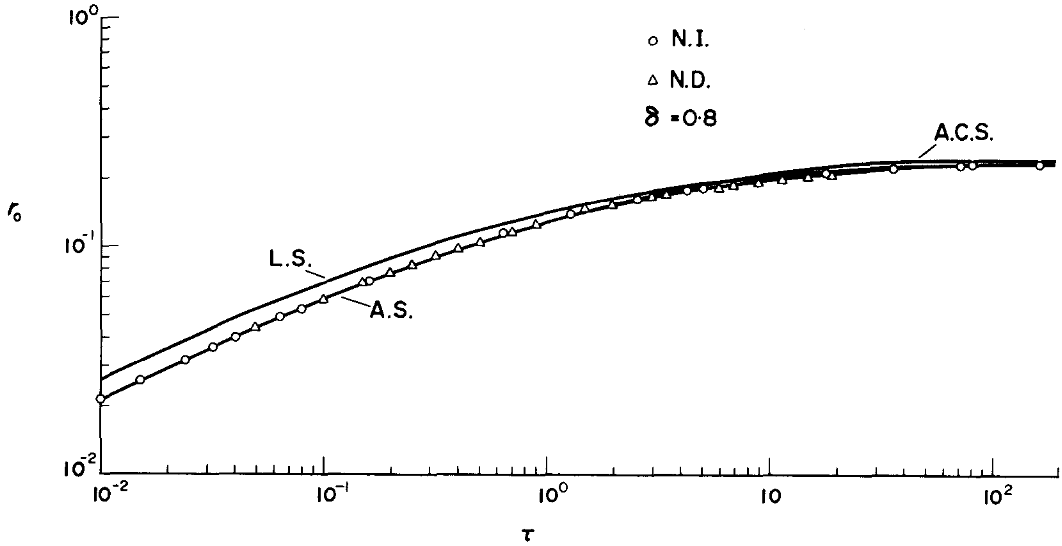

FIG. 2. Comparison between the approximate analytical solutions and the numerical solutions for $\delta = 0{\cdot}8$. ○: as in Fig. 1; △: as in Fig. 1.

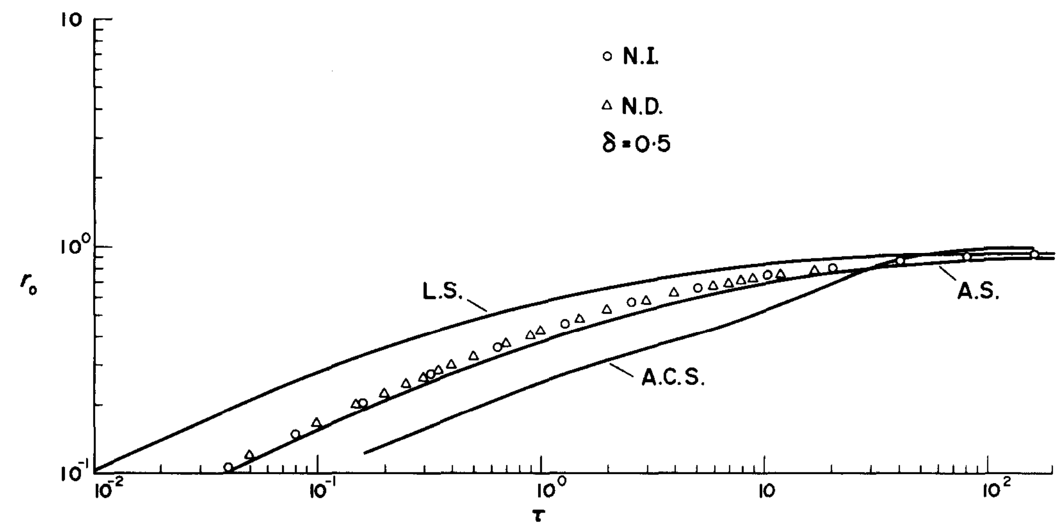

FIG. 3. Comparison between the approximate analytical solutions and the numerical solutions for $\delta = 0{\cdot}5$. ○: as in Fig. 1; △: as in Fig. 1.

where Γ is the gamma function and i^herfc means the repeated integral of the complementary error function. From the linearized solution (20)

$$r = \frac{1-\delta}{\delta}\left[\operatorname{erfc}\left(\frac{y}{2\sqrt{\tau}}\right) - \exp(y+\tau)\operatorname{erfc}\left(\tau + \frac{y}{2\sqrt{\tau}}\right)\right]. \quad (27)$$

Eventually equation (21) leads to the following expression:

$$r = \frac{1-\delta}{\delta}\left[\operatorname{erfc}\left(\frac{y}{2\sqrt{\tau}}\right) - \exp(\delta y + \delta^2\tau)\operatorname{erfc}\left(\delta\sqrt{\tau} + \frac{y}{2\sqrt{\tau}}\right)\right]. \quad (28)$$

ANALYSIS OF THE RESULTS

The accuracy of the numerical solution is very satisfactory. Comparison with the analytical solution in the case $\delta = 0$, shows that the numerical results are accurate up to at least four significant figures in the entire ranges of τ. Therefore it can be inferred that also for $\delta \neq 0$ the numerical results presented here are a valid test for the accuracy of the approximate solutions.

In the Figs. 2–4 the following approximate solutions are compared with the numerical solutions for values of δ equal to 0·8; 0·5; 0·01:

(i) The solution linearized with respect to $(1-\delta)/\delta$, (L.S.) equation (20)
(ii) The asymptotic solution (A.S.), valid for $\tau \to \infty$, equation (21)
(iii) The solution valid when the cross velocity V is almost constant (A.C.S.), accurate for small values of both δ and τ, equation (22)
(iv) The initial solution (I.S.) valid for small value of δ and for $0 \leqslant \tau \leqslant 100$, equation (23)

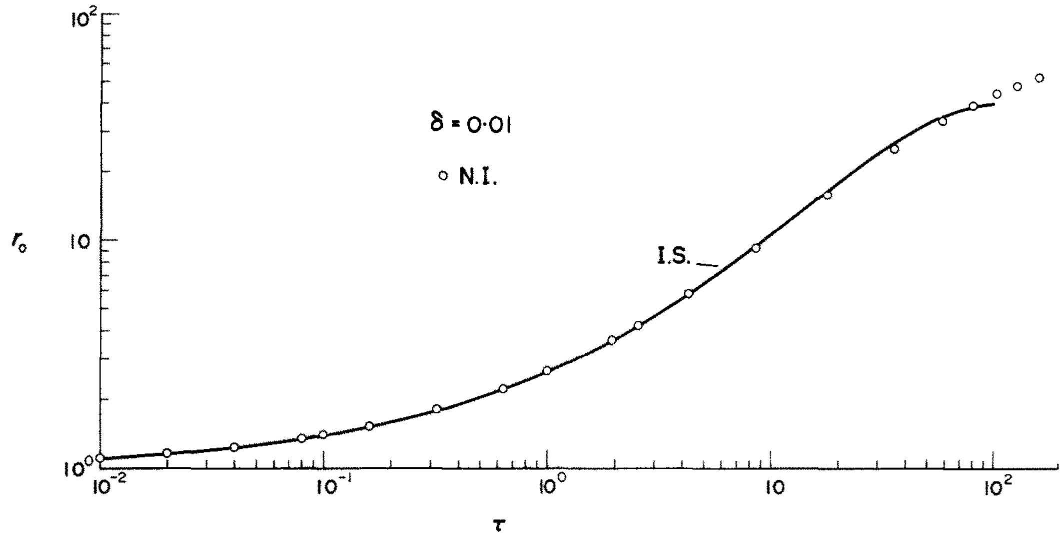

FIG. 4. Comparison between the approximate analytical solution and the numerical solution obtained by means of integral equation (10) for $\delta = 0{\cdot}01$. ○: as in Fig. 1.

Let $\bar{\varepsilon}$ be the highest percentage error in the entire τ field. When $\delta = 0{\cdot}8$ all the approximate solutions are satisfactory. In particular, the most accurate solution is the asymptotic one ($\bar{\varepsilon} = -0{\cdot}3$ per cent) whereas the others have $\bar{\varepsilon} = 1{\cdot}3$ per cent (A.C.S.) and $\bar{\varepsilon} = 1$ per cent (L.S.).

For $\delta = 0{\cdot}5$ the asymptotic solution is still very good ($\bar{\varepsilon} = -3$ per cent) while the other two solutions (A.C.S. and L.S.) show somewhat higher errors: $\bar{\varepsilon} = +11$ per cent and $\bar{\varepsilon} = -10$ per cent respectively.

For $\delta = 0{\cdot}01$ and $0 \leqslant \tau \leqslant 100$ the initial solution shows an error of less than 10 per cent.

For small values of τ a very simple expression for r_0 can be obtained from equation (13). Considering only two terms of the expansion in series one has

$$r_0 = (1-\delta)\left[2\left(\frac{\tau}{\pi}\right)^{1/2} + (1-3\delta)\frac{\tau}{2}\right] \qquad (29)$$

that furnishes very good values for $\tau \leqslant 1$. For $1 < \tau < 10$ equation (29) gives results not too different from the exact ones. For instance, in the case $\delta = 0$, for $\tau = 1, 4, 9$ one has the exact values 1·71; 5 and 9·96 while equation (29) gives 1·63, 4·26 and 7·89.

STEADY-STATE SOLUTION

For practical purposes an analysis of the asymptotic behaviour of the solution is very important. In this case (rejection coefficient equal one) the problem is very simple. In fact for $\tau \to \infty$ it is $r_0 \to (1-\delta)/\delta$ and $V \to 0$.

This result means that, from the industrial point of view, the steady-state situation for the case rejection coefficient equal to one is not of practical interest because the limiting flux value is zero. However the process might be of some interest from an industrial point of view at values of time not higher than specific τ_k at which the flux V is significantly different from zero.

More complex situations arise when the rejection coefficient is different from one and a discussion of this problem will be published very soon [2].

CONCLUDING REMARKS

In this note numerical and analytical solutions of the problem describing the separation of the components of a solution by an ideal semipermeable membrane were presented. A suitable application of the Laplace transform gave the governing equation in integral form.

Accurate numerical results are obtainable in a few minutes by an IBM 360/44 computer. A direct approach using the original differential equation would require comparatively longer computer times.

Approximate solutions were given in the whole range of τ and δ. In each case approximate values to less than 10 per cent were calculated.

Of particular interest is the solution for small values of δ and for $0 \leqslant \tau \leqslant 100$; in fact in [7] the absence of such a solution was noted. Moreover a very simple expression for the asymptotic time τ_k was obtained; where τ_k represents the value of the dimensionless time after which, the asymptotic value $(1-\delta)/\delta$ can be assumed instead of r_0, within a percentage error k.

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APPENDIX

Statement of equation (19).

By repeated applications of the Cauchy rule for the product of two series one has:

$$z^n = \sum_{i_n=0}^{\infty} A_{n,i_n}(\tau-s)^{i_n+n/2}$$

where

$$A_{n,i_n} = \sum_{i_{n-1}=0}^{i_n} A_{n-1,i_{n-1}} a_{i_n-i_{n-1}}\,;\; A_{1i1} = a_{i1}.$$

By application of the Cauchy rule, the two functions that must be integrated in equation (14) are:

$$\sum_{J=0}^{\infty}\sum_{m=0}^{J} G_m s^{m/2}\frac{(-1/4)^{J-m}}{(J-m)!}\sum_{i=0}^{\infty} A_{2(J-m),i}(\tau-s)^{J+i-m}$$

$$\sum_{J=1}^{\infty}\sum_{m=1}^{J} mc_m \frac{s^{(m/2)-1}}{2^{2(J-m+1)}}\frac{(-1)^{J-m}}{(J-m)![2(J-m)+1]} \times \sum_{i=0}^{\infty} A_{2(J-m)+1,i}\cdot(\tau-s)^{i+J-m+\frac{1}{2}}.$$

In this way all the integrals in equation (14) can be evaluated as

$$\pi^{-1/2}\int_0^{\tau} s^{\alpha}(\tau-s)^j\,\mathrm{d}s = I_{\alpha,j}\tau^{\alpha+j+1}$$

Therefore equation (14) becomes:

$$\sum_{i=1}^{\infty} c_i\tau^{i/2} = \sum_{J=0}^{\infty}\sum_{m=0}^{J}\frac{G_m}{(J-m)!(-4)^{J-m}}\sum_{i=0}^{\infty} I_{m/2,J-m+i-\frac{1}{2}} \times \sum_{k=0}^{\infty} c_{i,k}^{2(J-m)}\tau^{J+(1+k-m)/2}$$
$$+2\sum_{J=1}^{\infty}\sum_{m=1}^{J}\frac{m}{(-4)^{J-m+1}}\frac{c_m}{(J-m)![2(J-m)+1]} \times \sum_{i=0}^{\infty} I_{(m/2)-1,J-n+i+\frac{1}{2}}\sum_{k=0}^{\infty} c_{i,k}^{2(J-m)+1}\tau^{J+(k/2)-(m/2)}$$

To determine the unknown quantities c_i, the coefficients of the same powers of τ are equated.

In this way one has the equation (19).

SOLUTIONS ANALYTIQUES ET NUMERIQUES POUR LA POLARISATION PAR CONCENTRATION EN HYPERFILTRATION EN L'ABSENCE D'ECOULEMENT AXIAL

Résumé—On étudie le problème instationnaire, non-linéaire et unidimensionnel qui décrit la séparation des composants d'une solution à travers une membrane idéale semi-perméable.

Au moyen d'une transformation de Laplace convenable portant sur les variables d'espace, l'équation aux dérivées partielles non-linéaire qui gouverne le système a été transformée en une équation intégrale.

Des solutions numériques et des solutions analytiques approchées de l'équation intégrale fondamentale sont présentées.

Les résultats obtenus à l'aide de solutions analytiques approchées ont été comparés aux solutions numériques exactes qui sont trouvées en accord satisfaisant.

NUMERISCHE UND ANALYTISCHE LÖSUNGSANSÄTZE FÜR KONZENTRATIONS-POLARISIERUNG DURCH HYPERFILTRATION OHNE AXIALE STRÖMUNG

Zusammenfassung—Für den eindimensionalen Fall wird der Vorgang der instationären Trennung von Komponenten einer Lösung durch eine ideal-semipermeable Membran untersucht.

Vermittels geeigneter Anwendung der Laplace-Transformation wird die für das System geltende partielle Differentialgleichung in eine Integralgleichung umgeformt.

Für die Integralgleichung werden numerische und analytische Näherungslösungen angegeben. Die Ergebnisse der analytischen Näherungslösung werden mit den Ergebnissen der numerischen Auswertung verglichen, wobei befriedigende Übereinstimmung feststellbar ist.

ЧИСЛЕННЫЕ И АНАЛИТИЧЕСКИЕ РЕШЕНИЯ КОНЦЕНТРАЦИОННОЙ ПОЛЯРИЗАЦИИ ПРИ ГИПЕРФИЛЬТРАЦИИ БЕЗ АКСИАЛЬНОГО ПОТОКА

Аннотация — Исследуется нелинейная одномерная нестационарная задача, описывающая разделения компонент раствора с помощью идеальной полупроницаемой мембраны. Описывающее систему нелинейное дифференциальное уравнение в частных производных преобразовано в интегральное уравнение с помощью соответствующего использования преобразования Лапласа относительно пространственной координаты.

Представлены численные и приближенные аналитические решения основного интегрального уравнения.

Проведено сравнение и обнаружено удовлетворительное согласование между полученными приближенными аналитическими решениями и точными численными решениями.